An Implicit Hierarchical Fixed Point Approach to General Variational Inequalities in Hilbert Spaces

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Abstract. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$, $V, T : C \to C$ be nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$ where $\mathrm{Fix}(T)$ denotes the fixed point set of T, and $f : C \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1)$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. For each $s, t \in (0, 1)$, let $x_{s,t}$ be a unique solution of the fixed point equation $x_{s,t} = P_C[s\gamma f(x_{s,t}) + (I - s\mu F)(tV + (1-t)T)x_{s,t}]$. We derive the following conclusions on the behavior of the net $\{x_{s,t}\}$ along the curve $t = t(s)$:

(i) if $t(s) = O(s)$, as $s \to 0$, then $x_{s,t(s)} \to z_\infty$ strongly, which is the unique solution of the variational inequality of finding $z_\infty \in \mathrm{Fix}(T)$ such that

$$\langle [(\mu F - \gamma f) + l(I - V)]z_\infty, x - z_\infty \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

(ii) if $t(s)/s \to \infty$, as $s \to 0$, then $x_{s,t(s)} \to x_\infty$ strongly, which is the unique solution of some hierarchical variational inequality problem.

Keywords: Implicit method; General variational inequality; Hierarchical fixed point; Nonexpansive mapping; Projection; Demiclosedness principle

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. Let $T : C \to C$ be a nonexpansive mapping; namely, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by the set $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$. It is well known that if $\mathrm{Fix}(T) \neq \emptyset$ then $\mathrm{Fix}(T)$ is closed and convex. Given nonexpansive mapping $V : C \to C$, consider the variational inequality (for short, VI) of finding hierarchically a fixed point $x^* \in \mathrm{Fix}(T)$ of T with respect to V such that

$$\langle (I - V)x^*, y - x^* \rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T). \tag{1.1}$$

Equivalently, $x^* = P_{\mathrm{Fix}(T)}Vx^*$; that is, x^* is a fixed point of the nonexpansive mapping $P_{\mathrm{Fix}(T)}V$, where P_K denotes the metric projection from H onto a nonempty closed convex subset K of H. Let S denote the solution set of the the VI (1.1) and assume throughout the rest of this paper that $S \neq \emptyset$. It is easy to see that $S = \mathrm{Fix}(P_{\mathrm{Fix}(T)}V)$. The VI (1.1) covers several topics investigated in the literature; see, e.g., [1,3,5,6,8,11,12]. Related iterative methods for solving fixed point problems, variational inequalities and optimization problems can also be found in [14-26].

Let $f : C \to C$ be a ρ-contraction and define, for $s, t \in (0, 1)$, two mappings W_t and $f_{s,t}$ by

$$W_t = tV + (1 - t)T \quad \text{and} \quad f_{s,t} = sf + (1 - s)W_t.$$

It is easy to verify that W_t is nonexpansive and $f_{s,t}$ is a $[1 - (1 - \rho)s]$-contraction.

Let $x_{s,t}$ be the unique fixed point of $f_{s,t}$, that is, the unique solution of the fixed point equation

$$x_{s,t} = sf(x_{s,t}) + (1 - s)W_t x_{s,t}. \tag{1.2}$$

Moudafi and Mainge [7] initiated the investigation of the iterated behavior of the net $\{x_{s,t}\}$ as $s \to 0$ firstly and $t \to 0$ secondly. They made the following assumptions:

(A1) for each $t \in (0, 1)$, the fixed point set $\mathrm{Fix}(W_t)$ of W_t is nonempty and the set

$$\{\mathrm{Fix}(W_t) : 0 < t < 1\} = \bigcup_{t \in (0,1)} \mathrm{Fix}(W_t)$$

is bounded;

(A2) $\emptyset \neq S \subset \|\cdot\| - \liminf_{t \to 0} \mathrm{Fix}(W_t) := \{z : \exists z_t \in \mathrm{Fix}(W_t) \text{ such that } z_t \to z\}$.

Moudafi and Mainge [7] (see also [9]) proved that, for each fixed $t \in (0, 1)$, as $s \to 0$, $x_{s,t} \to x_t$; moreover, as $t \to 0$, $x_t \rightharpoonup x_\infty$ which is the unique solution of the variational inequality of finding $x_\infty \in S$ such that

$$\langle (I - f)x_\infty, x - x_\infty \rangle \geq 0, \quad \forall x \in S. \tag{1.3}$$

The following theorem, due to Xu [10], improves Moudafi and Mainge's result since it shows that $\{x_t\}$ actually strongly converges to x_∞. Moreover, it does not need the boundedness assumption of the set $\bigcup_{t \in (0,1)} \mathrm{Fix}(W_t)$.

Theorem 1.1 ([10, Theorem 3.2]). Let the above assumption (A2) hold. Assume also that, for each $t \in (0,1)$, $\mathrm{Fix}(W_t)$ is nonempty (but not necessarily bounded). Then the strong $\lim_{s\to 0} x_{s,t} =: x_t$ exists for each $t \in (0,1)$. Moreover, the strong $\lim_{t\to 0} x_t =: x_\infty$ exists and solves the VI (1.3). Hence, for each null sequence $\{s_n\}$ in $(0,1)$, there is another null sequence $\{t_n\}$ in $(0,1)$ such that x_{s_n,t_n}, as $n \to \infty$.

In [7,10], the authors stated the problem of the convergence of $\{x_{s,t}\}$ when $(s,t) \to (0,0)$ jointly. Very recently, Cianciaruso, Colao, Muglia and Xu [13] further investigated the behavior of the net $\{x_{s,t}\}$ along the curve $t = t(s)$ and their results point to a negative answer to this problem.

Theorem 1.2 ([13, Theorem 2.1]). Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $V, T : C \to C$ be nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$. Let $f : C \to C$ be a ρ-contraction with $\rho \in [0,1)$. Assume that $t_s = O(s)$, as $s \to 0$, and let $l = \limsup_{s\to 0}(t_s/s)$. Then the net $\{x_{s,t_s}\}_{s\in(0,1)}$ defined by

$$x_{s,t_s} = sf(x_{s,t_s}) + (1-s)W_{t_s}x_{s,t_s}, \tag{1.4}$$

strongly converges to $z_\infty \in \mathrm{Fix}(T)$ which is the unique solution of the variational inequality of finding $z_\infty \in \mathrm{Fix}(T)$ such that

$$\langle [(I-f) + l(I-V)]z_\infty, x - z_\infty\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.5}$$

Theorem 1.3 ([13, Theorem 3.1]). Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Assume that $V, T : C \to C$ are nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$ and $f : C \to C$ is a ρ-contraction with $\rho \in [0,1)$. Assume the condition (A2) holds. Let $t_s = t(s)$ satisfy $\lim_{s\to 0} t_s/s = \infty$. Then the net $\{x_{s,t_s}\}_{s\in(0,1)}$ defined by

$$x_{s,t_s} = sf(x_{s,t_s}) + (1-s)W_{t_s}x_{s,t_s},$$

strongly converges to $x_\infty \in S$ which is the unique solution of the VI (1.3).

On the other hand, let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$, and let $T : H \to H$ be nonexpansive such that $\mathrm{Fix}(T) \neq \emptyset$. In 2001, Yamada [11] introduced the so-called hybrid steepest-descent method for solving the variational inequality problem: finding $\tilde{x} \in \mathrm{Fix}(T)$ such that

$$\langle F\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

This method generates a sequence $\{x_n\}$ via the following iterative scheme:

$$x_{n+1} = Tx_n - \lambda_{n+1}\mu F(Tx_n), \quad \forall n \geq 0, \tag{1.6}$$

where $0 < \mu < 2\eta/\kappa^2$, the initial guess $x_0 \in H$ is arbitrary and the sequence $\{\lambda_n\}$ in $(0,1)$ satisfies the conditions:

$$\lambda_n \to 0, \quad \sum_{n=0}^{\infty} \lambda_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

A key fact in Yamada's argument is that, for small enough $\lambda > 0$, the mapping

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in H$$

is a contraction, due to the κ-Lipschitz continuity and η-strong monotonicity of F.

In this paper, let C be a nonempty closed convex subset of a real Hilbert space H. Assume $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$, $f : C \to H$ is a ρ-contraction with coefficient $\rho \in [0, 1)$ and $T, V : C \to C$ are nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Consider the hierarchical variational inequality problem (for short, HVIP):

VI (a): finding $z^* \in \mathrm{Fix}(T)$ such that $\langle (I - V)z^*, z - z^* \rangle \geq 0,\ \forall z \in \mathrm{Fix}(T)$;

VI (b): finding $x^* \in S$ such that $\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0,\ \forall x \in S$.

Here S denotes the nonempty solution set of the VI (a).

Motivated and inspired by the above hybrid steepest-descent method and hierarchical fixed point approximation method, we define, for each $s, t \in (0, 1)$, two mappings W_t and $f_{s,t}$ by

$$W_t = tV + (1 - t)T \quad \text{and} \quad f_{s,t} = P_C[s\gamma f + (I - s\mu F)W_t].$$

It is easy to see that W_t is a nonexpansive self-mapping on C. Moreover, utilizing Lemma 2.5 in Section 2, we can see that $f_{s,t}$ is a $(1 - (\tau - \gamma\rho)s)$-contraction. Indeed, observe that

$$\begin{aligned}
\|f_{s,t}(x) - f_{s,t}(y)\| &= \|P_C[s\gamma f(x) + (I - s\mu F)W_t x] - P_C[s\gamma f(y) + (I - s\mu F)W_t y]\| \\
&\leq \|[s\gamma f(x) + (I - s\mu F)W_t x] - [s\gamma f(y) + (I - s\mu F)W_t y]\| \\
&\leq s\gamma\|f(x) - f(y)\| + \|(I - s\mu F)W_t x - (I - s\mu F)W_t y\| \\
&\leq s\gamma\rho\|x - y\| + (1 - s\tau)\|x - y\| \\
&= (1 - (\tau - \gamma\rho)s)\|x - y\|.
\end{aligned}$$

Let $x_{s,t}$ be the unique fixed point of $f_{s,t}$ in C, that is, the unique solution of the fixed point equation

$$x_{s,t} = P_C[s\gamma f(x_{s,t}) + (I - s\mu F)W_t(x_{s,t})]. \tag{1.7}$$

We investigate the behavior of the net $\{x_{s,t}\}$ (generated by (1.7)) along the curve $t = t(s)$ and our results give a negative answer to the problem put forth in [7,10]. Specifically, we derive the following conclusions:

(i) if $t(s) = O(s)$, as $s \to 0$, then $x_{s,t(s)} \to z_\infty \in \mathrm{Fix}(T)$, which is the unique solution of the variational inequality of finding $z_\infty \in \mathrm{Fix}(T)$ such that

$$\langle [(\mu F - \gamma f) + l(I - V)]z_\infty, x - z_\infty \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

(ii) if $t(s)/s \to \infty$, as $s \to 0$, then $x_{s,t(s)} \to x_\infty \in S$, which is the unique solution of the VI (b).

In particular, if we put $\mu = 1,\ F = I$ and $\gamma = \tau = 1$ and let f be a contractive self-mapping on C with coefficient $\rho \in [0, 1)$, then our results reduce to the above Theorems 1.2 and 1.3, respectively. There is no doubt that our results cover the Theorems 1.2 and 1.3 as special cases, respectively. In the meantime, our results also extend and improve Xu's Theorem 3.2 [10].

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that the metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Lemma 2.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$.

(i) That $z = P_C x$ if and only if there holds the relation:

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

(ii) That $z = P_C x$ if and only if there holds the relation:

$$\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C.$$

(iii) There holds the relation

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Consequently, P_C is nonexpansive and monotone.

Lemma 2.2 (See [2, Demiclosedness principle]). Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.

The following lemmas are not difficult to prove.

Lemma 2.3. Let C be a nonempty closed convex subset of a real Hilbert space H, $f : C \to H$ a ρ-contraction with coefficient $\rho \in [0, 1)$, and $F : C \to H$ a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$. Then for $0 \leq \gamma\rho < \mu\eta$,

$$\langle x - y, (\mu F - \gamma f)x - (\mu F - \gamma f)y\rangle \geq (\mu\eta - \gamma\rho)\|x - y\|^2, \quad \forall x, y \in C.$$

That is, $\mu F - \gamma f$ is strongly monotone with constant $\mu\eta - \gamma\rho$.

Lemma 2.4. There holds the following inequality in a real Hilbert space H:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

The following lemma plays a key role in proving strong convergence of our implicit hybrid method.

Lemma 2.5 (See [8, Lemma 3.1]). Let λ be a number in $(0,1]$ and let $\mu > 0$. Let $F : C \to H$ be an operator on C such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Associating with a nonexpansive mapping $T : C \to C$, define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C.$$

Then T^λ is a contraction provided $\mu < 2\eta/\kappa^2$, that is,

$$\|T^\lambda x - T^\lambda y\| \le (1-\lambda\tau)\|x-y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1]$.

Remark 2.1. Put $F = I$, where I is the identity operator of H. Then $\kappa = \eta = 1$ and hence $\mu < 2\eta/\kappa^2 = 2$. Also, put $\mu = 1$. Then it is easy to see that

$$\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} = 1.$$

In particular, whenever $\lambda > 0$, we have $T^\lambda x := Tx - \lambda\mu F(Tx) = (1-\lambda)Tx$.

3. On Convergence of $\{x_{s,t}\}_{s,t\in(0,1)}$

In this section we study the convergence of the net $\{x_{s,t}\}$ along the curve $t = t(s) =: t_s$, where $t_s = O(s)$, as $s \to 0$.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$, $V, T : C \to C$ be nonexpansive mappings with $\mathrm{Fix}(T) \ne \emptyset$, and $f : C \to H$ be a ρ-contraction with coefficient $\rho \in [0,1)$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \le \tau$, where $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$. Assume that $t_s = O(s)$, as $s \to 0$, and let $l = \limsup_{s\to 0}(t_s/s)$. Then the net $\{x_{s,t_s}\}_{s\in(0,1)}$ defined by

$$x_{s,t_s} = P_C[s\gamma f(x_{s,t_s}) + (I - s\mu F)W_{t_s}x_{s,t_s}], \tag{3.1}$$

where $W_{t_s} = t_sV + (1-t_s)T$, strongly converges to a fixed point z_∞ of T which is the unique solution of the variational inequality of finding $z_\infty \in \mathrm{Fix}(T)$ such that

$$\langle [(\mu F - \gamma f) + l(I-V)]z_\infty, x - z_\infty\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T). \tag{3.2}$$

Proof. First, let us show that the VI (3.2) has a unique solution. Indeed, it is sufficient to show that the operator $(\mu F-\gamma f)+l(I-V)$ is strongly monotone. Observe that

$$\begin{aligned}\mu\eta\geq\tau\quad&\Leftrightarrow\quad \mu\eta\geq 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}\\&\Leftrightarrow\quad \sqrt{1-\mu(2\eta-\mu\kappa^2)}\geq 1-\mu\eta\\&\Leftrightarrow\quad 1-2\mu\eta+\mu^2\kappa^2\geq 1-2\mu\eta+\mu^2\eta^2\\&\Leftrightarrow\quad \kappa^2\geq\eta^2\\&\Leftrightarrow\quad \kappa\geq\eta,\end{aligned}$$

and

$$\begin{aligned}&\langle[(\mu F-\gamma f)+l(I-V)]x-[(\mu F-\gamma f)+l(I-V)]y,x-y\rangle\\&=\langle(\mu F-\gamma f)x-(\mu F-\gamma f)y,x-y\rangle+l\langle(I-V)x-(I-V)y,x-y\rangle\\&\geq\langle(\mu F-\gamma f)x-(\mu F-\gamma f)y,x-y\rangle\\&\geq(\mu\eta-\gamma\rho)\|x-y\|^2,\quad \forall x,y\in C.\end{aligned}$$

Since

$$0\leq\gamma\rho<\gamma\leq\tau\leq\mu\eta,$$

it follows that $(\mu F-\gamma f)+l(I-V)$ is strongly monotone with constant $\mu\eta-\gamma\rho>0$. So the variational inequality (3.2) has only one solution. Below we use $z_\infty\in\mathrm{Fix}(T)$ to denote the unique solution of VI (3.2).

The remainder of the proof is divided into two steps.

The first step is to prove that the net $\{x_{s,t_s}\}_{s\in(0,1)}$ is bounded. Indeed, set

$$y_{s,t_s}=s\gamma f(x_{s,t_s})+(I-s\mu F)W_{t_s}x_{s,t_s},$$

where $W_{t_s}=t_sV+(1-t_s)T$. Take a fixed $p\in\mathrm{Fix}(T)$ arbitrarily. Then from (3.1) we obtain that $x_{s,t_s}=P_Cy_{s,t_s}$ and

$$\begin{aligned}y_{s,t_s}-p\ &=s\gamma f(x_{s,t_s})+(I-s\mu F)W_{t_s}x_{s,t_s}-p\\&=s(\gamma f(x_{s,t_s})-\mu FW_{t_s}p)+(I-s\mu F)W_{t_s}x_{s,t_s}-(I-s\mu F)W_{t_s}p+W_{t_s}p-p\\&=s\gamma(f(x_{s,t_s})-f(p))+s(\gamma f(p)-\mu FW_{t_s}p)+(I-s\mu F)W_{t_s}x_{s,t_s}-(I-s\mu F)W_{t_s}p\\&\quad+t_s(V-I)p.\end{aligned}$$

Since P_C is the metric projection from H onto C, utilizing Lemma 2.1, we have

$$\langle P_Cy_{s,t_s}-y_{s,t_s},P_Cy_{s,t_s}-p\rangle\leq 0.$$

Thus utilizing Lemma 2.5 we get

$$\begin{aligned}
\|x_{s,t_s}-p\|^2 &= \langle P_C y_{s,t_s}-y_{s,t_s}, P_C y_{s,t_s}-p\rangle + \langle y_{s,t_s}-p, x_{s,t_s}-p\rangle \\
&\le \langle y_{s,t_s}-p, x_{s,t_s}-p\rangle \\
&= s\gamma\langle f(x_{s,t_s})-f(p), x_{s,t_s}-p\rangle + s\langle \gamma f(p)-\mu FW_{t_s}p, x_{s,t_s}-p\rangle \\
&\quad + \langle (I-s\mu F)W_{t_s}x_{s,t_s}-(I-s\mu F)W_{t_s}p, x_{s,t_s}-p\rangle + t_s\langle (V-I)p, x_{s,t_s}-p\rangle \\
&\le s\gamma\|f(x_{s,t_s})-f(p)\|\|x_{s,t_s}-p\| + s\langle \gamma f(p)-\mu FW_{t_s}p, x_{s,t_s}-p\rangle \\
&\quad + \|(I-s\mu F)W_{t_s}x_{s,t_s}-(I-s\mu F)W_{t_s}p\|\|x_{s,t_s}-p\| + t_s\langle (V-I)p, x_{s,t_s}-p\rangle \\
&\le s\gamma\rho\|x_{s,t_s}-p\|^2 + s\langle \gamma f(p)-\mu FW_{t_s}p, x_{s,t_s}-p\rangle + (1-s\tau)\|x_{s,t_s}-p\|^2 \\
&\quad + t_s\langle (V-I)p, x_{s,t_s}-p\rangle \\
&= (1-s(\tau-\gamma\rho))\|x_{s,t_s}-p\|^2 + s\langle \gamma f(p)-\mu FW_{t_s}p, x_{s,t_s}-p\rangle \\
&\quad + t_s\langle (V-I)p, x_{s,t_s}-p\rangle,
\end{aligned}$$

which hence implies that

$$\|x_{s,t_s}-p\|^2 \le \frac{1}{\tau-\gamma\rho}[\langle \gamma f(p)-\mu FW_{t_s}p, x_{s,t_s}-p\rangle + \frac{t_s}{s}\langle (V-I)p, x_{s,t_s}-p\rangle]. \tag{3.3}$$

In particular,

$$\|x_{s,t_s}-p\| \le \frac{1}{\tau-\gamma\rho}[\|\gamma f(p)-\mu FW_{t_s}p\| + \frac{t_s}{s}\|(V-I)p\|]. \tag{3.4}$$

Note

$$\|W_{t_s}p-p\| = t_s\|(V-I)p\| \le \|(V-I)p\|. \tag{3.5}$$

Hence we have

$$\|W_{t_s}p\| \le \|p\| + \|(V-I)p\|.$$

Since $t_s = O(s)$, as $s \to 0$, (3.4) implies the boundedness of $\{x_{s,t_s}\}$ and the first step is proved.

The second step is to prove that the net $x_{s,t_s} \to z_\infty \in \mathrm{Fix}(T)$, as $s \to 0$, where z_∞ is the unique solution of the VI (3.2). Indeed, observe that

$$\begin{aligned}
\|x_{s,t_s}-Tx_{s,t_s}\| &\le s\gamma\|f(x_{s,t_s})\| + \|(I-s\mu F)W_{t_s}x_{s,t_s}-Tx_{s,t_s}\| \\
&\le s\gamma\|f(x_{s,t_s})\| + s\mu\|FW_{t_s}x_{s,t_s}\| + \|W_{t_s}x_{s,t_s}-Tx_{s,t_s}\| \\
&= s\gamma\|f(x_{s,t_s})\| + s\mu\|FW_{t_s}x_{s,t_s}\| + t_s\|Vx_{s,t_s}-Tx_{s,t_s}\|.
\end{aligned} \tag{3.6}$$

From (3.5) it follows that

$$\begin{aligned}
\|FW_{t_s}x_{s,t_s}-Fp\| &= \|FW_{t_s}x_{s,t_s}-FW_{t_s}p+FW_{t_s}p-Fp\| \\
&\le \kappa(\|x_{s,t_s}-p\| + \|W_{t_s}p-p\|) \\
&\le \kappa(\|x_{s,t_s}-p\| + \|(V-I)p\|).
\end{aligned} \tag{3.7}$$

Since $\{x_{s,t_s}\}$ is bounded when $s \to 0$, (3.7) implies the boundedness of $\{FW_{t_s}x_{s,t_s}\}$. Consequently, noticing that $\{x_{s,t_s}\}$ and $\{FW_{t_s}x_{s,t_s}\}$ are bounded when $s \to 0$ (hence $t_s \to 0$), we conclude from (3.6) that

$$\|x_{s,t_s}-Tx_{s,t_s}\| \to 0. \tag{3.8}$$

We now claim that $\{x_{s,t_s}\}_{s\in(0,1)}$ is relatively compact as $s \to 0$ in the norm topology. To see this, assume $\{s_n\}$ is a null sequence in $(0,1)$. Without loss of generality, we may assume that $x_{s_n,t_{s_n}} \rightharpoonup \widehat{x}$ which implies from (3.8) and Lemma 2.2 that $\widehat{x} \in \mathrm{Fix}(T)$. It is clear that $FW_{t_{s_n}}\widehat{x} = F(t_{s_n}V\widehat{x} + (1-t_{s_n})\widehat{x}) \to F\widehat{x}$ as $n \to \infty$. This implies that as $n \to \infty$,

$$\begin{aligned}&|\langle \gamma f(\widehat{x}) - \mu FW_{t_{s_n}}\widehat{x}, x_{s_n,t_{s_n}} - \widehat{x}\rangle|\\&= |\langle \gamma f(\widehat{x}) - \mu F\widehat{x}, x_{s_n,t_{s_n}} - \widehat{x}\rangle + \langle \mu F\widehat{x} - \mu FW_{t_{s_n}}\widehat{x}, x_{s_n,t_{s_n}} - \widehat{x}\rangle|\\&\le |\langle \gamma f(\widehat{x}) - \mu F\widehat{x}, x_{s_n,t_{s_n}} - \widehat{x}\rangle| + \mu\|F\widehat{x} - FW_{t_{s_n}}\widehat{x}\|\|x_{s_n,t_{s_n}} - \widehat{x}\| \to 0.\end{aligned}$$

We thus immediately get from (3.3) that $x_{s_n,t_{s_n}} \to \widehat{x}$.

We next further claim that $\widehat{x} = z_\infty$, the unique solution of the VI (3.2), which then completes the proof. Indeed, observe that

$$(\mu F - \gamma f)x_{s,t_s} = \frac{1}{s}(P_C y_{s,t_s} - y_{s,t_s}) - \frac{1}{s}(I - W_{t_s})x_{s,t_s} + \mu(Fx_{s,t_s} - FW_{t_s}x_{s,t_s}).$$

Hence, utilizing Lemma 2.1 we deduce from the monotonicity of $\mu F - \gamma f$ and $I - W_{t_s}$ that for any fixed $p \in \mathrm{Fix}(T)$

$$\begin{aligned}&\langle(\mu F - \gamma f)p, x_{s,t_s} - p\rangle \le \langle(\mu F - \gamma f)x_{s,t_s}, x_{s,t_s} - p\rangle\\&= \tfrac{1}{s}\langle P_C y_{s,t_s} - y_{s,t_s}, P_C y_{s,t_s} - p\rangle - \tfrac{1}{s}\langle(I - W_{t_s})x_{s,t_s}, x_{s,t_s} - p\rangle\\&\quad + \mu\langle Fx_{s,t_s} - FW_{t_s}x_{s,t_s}, x_{s,t_s} - p\rangle\\&\le -\tfrac{1}{s}\langle(I - W_{t_s})x_{s,t_s}, x_{s,t_s} - p\rangle + \mu\langle Fx_{s,t_s} - FW_{t_s}x_{s,t_s}, x_{s,t_s} - p\rangle\\&= -\tfrac{1}{s}\langle(I - W_{t_s})x_{s,t_s} - (I - W_{t_s})p, x_{s,t_s} - p\rangle - \tfrac{1}{s}\langle(I - W_{t_s})p, x_{s,t_s} - p\rangle\\&\quad + \mu\langle Fx_{s,t_s} - FW_{t_s}x_{s,t_s}, x_{s,t_s} - p\rangle\\&\le -\tfrac{1}{s}\langle(I - W_{t_s})p, x_{s,t_s} - p\rangle + \mu\langle Fx_{s,t_s} - FW_{t_s}x_{s,t_s}, x_{s,t_s} - p\rangle\\&= \tfrac{t_s}{s}\langle(V - I)p, x_{s,t_s} - p\rangle + \mu\langle Fx_{s,t_s} - FW_{t_s}x_{s,t_s}, x_{s,t_s} - p\rangle.\end{aligned} \tag{3.9}$$

Now since $x_{s_n,t_{s_n}} \to \widehat{x}$, we have

$$Fx_{s_n,t_{s_n}} - FW_{t_{s_n}}x_{s_n,t_{s_n}} = Fx_{s_n,t_{s_n}} - F[t_{s_n}Vx_{s_n,t_{s_n}} + (1 - t_{s_n})Tx_{s_n,t_{s_n}}] \to F\widehat{x} - F\widehat{x} = 0.$$

So, putting $s = s_n$ and $t = t_{s_n}$ in (3.9) and letting $n \to \infty$, we immediately conclude that

$$\langle(\mu F - \gamma f)p, \widehat{x} - p\rangle \le l\langle(V - I)p, \widehat{x} - p\rangle, \quad \forall p \in \mathrm{Fix}(T).$$

That is,

$$\langle[(\mu F - \gamma f) + l(I - V)]p, \widehat{x} - p\rangle \le 0, \quad \forall p \in \mathrm{Fix}(T).$$

Upon replacing the p in the last inequality with $\widehat{x} + \alpha(q - \widehat{x}) \in \mathrm{Fix}(T)$, where $\alpha \in (0,1)$ and $q \in \mathrm{Fix}(T)$, we get

$$\langle[(\mu F - \gamma f) + l(I - V)](\widehat{x} + \alpha(q - \widehat{x})), \widehat{x} - q\rangle \le 0.$$

Letting $\alpha \to 0$, we obtain the VI

$$\langle[(\mu F - \gamma f) + l(I - V)]\widehat{x}, \widehat{x} - q\rangle \le 0, \quad \forall q \in \mathrm{Fix}(T).$$

We immediately see that $\widehat{x}$ satisfies the VI (3.2) and therefore we must have $\widehat{x} = z_\infty$ since z_∞ is the unique solution of (3.2). □

Remark 3.1. (i) If $t_s = o(s)$ (that is, $l = 0$), then the above argument shows that the net $\{x_{s,t_s}\}$ actually converges in norm to the unique solution of the variational inequality of finding $x_\infty \in \mathrm{Fix}(T)$ such that

$$\langle (\mu F - \gamma f)x_\infty, p - x_\infty \rangle \geq 0, \quad \forall p \in \mathrm{Fix}(T), \tag{3.10}$$

which is also the unique fixed point of the contraction $P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)$, $x_\infty = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)x_\infty$. In particular, if $\mu = 1$, $F = I$, $\gamma = \tau = 1$ and f is a ρ-contractive self-mapping on C, then this is Theorem 3.2 in Xu [10].

(ii) The net $\{x_{s,t}\}_{s,t\in(0,1)}$ does not converge, in general, as $(s,t) \to (0,0)$ jointly, to the unique solution $x_\infty \in S$ of the VI (b) in Section 1. As a matter of fact, if $\{x_{s,t}\}_{s,t\in(0,1)}$ converges to x_∞ jointly as $(s,t) \to (0,0)$, then (by (3.10) we would have the relation and the VI (b))

$$x_\infty = P_S(I - \mu F + \gamma f)x_\infty = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)x_\infty$$

for all ρ-contraction f. In particular, if $\mu = 1$, $F = I$ and $\gamma = \tau = 1$, then $x_\infty = P_S f(x_\infty) = P_{\mathrm{Fix}(T)} f(x_\infty)$ for all ρ-contraction f. This implies that $S = \mathrm{Fix}(T)$ which is not true, in general.

(iii) Consider the case of $l > 0$. If x_∞, the unique solution of (3.10), belongs to S, then, clearly, $x_\infty = z_\infty$. If $x_\infty \notin S$, the following example shows that there are, in general, no links among z_∞, S and x_∞. Take

$$C = [0,1], \quad \mu = 1, \quad F = I, \quad \gamma = \tau = 1, \quad T = I, \quad f(x) = \frac{x}{2}, \quad Vx = 1 - x, \quad l = 1.$$

Then $\mathrm{Fix}(T) = [0,1]$. Moreover, the unique solution x_∞ of the variational inequality of finding $x_\infty \in [0,1]$ such that

$$\langle (\mu F - \gamma f)x_\infty, z - x_\infty \rangle \geq 0, \quad \forall z \in [0,1],$$
$$\text{(that is,} \quad \langle (I - f)x_\infty, z - x_\infty \rangle \geq 0, \quad \forall z \in [0,1])$$

is $x_\infty = 0$; the unique solution z_∞ of the variational inequality of finding $z_\infty \in [0,1]$ such that

$$\langle [(\mu F - \gamma f) + l(I - V)]z_\infty, z - z_\infty \rangle \geq 0, \quad \forall z \in [0,1],$$

$$\text{(that is,} \quad \langle [(I - f) + (I - V)]z_\infty, z - z_\infty \rangle \geq 0, \quad \forall z \in [0,1])$$

is $z_\infty = \frac{2}{5}$, and the set S of solutions to the variational inequality of finding $x \in [0,1]$ such that

$$\langle (I - V)x, z - x \rangle \geq 0, \quad \forall z \in [0,1],$$

is the singleton $\{\frac{1}{2}\}$.

Remark 3.2. Compared with Theorem 2.1 of Cianciaruso, Colao, Muglia and Xu [13], our Theorem 3.1 improves and extends their Theorem 2.1 [13] in the following aspects:

(i) The (self) contraction $f : C \to C$ in [13, Theorem 2.1] is extended to the case of (possibly nonself) contraction $f : C \to H$ on a nonempty closed convex subset C of H.

(ii) The convex combination of (self) contraction f and nonexpansive mapping W_{t_s} in the implicit scheme (2.1) of Theorem 2.1 [13] is extended to the linear combination of (possibly nonself) contraction f and hybrid steepest-descent method involving W_{t_s}.

(iii) In order to guarantee that the net $\{x_{s,t_s}\}$ generated by the implicit scheme still lies in C, the implicit scheme (2.1) in [13, Theorem 2.1] is extended to develop our new implicit scheme (3.1) by virtue of the projection method.

(iv) The new technique of argument is applied to deriving our Theorem 3.1. For instance, the characteristic properties (Lemma 2.4) of the metric projection play a key role in proving the strong convergence of the net $\{x_{s,t_s}\}_{s\in(0,1)}$ in our Theorem 3.1.

(v) If we put $\mu = 1$, $F = I$ and $\gamma = \tau = 1$ and let f be a contractive self-mapping on C with coefficient $\rho \in [0, 1)$, then our Theorem 3.1 reduces to Theorem 2.1 [13]. Thus, our Theorem 3.1 covers Theorem 2.1 [13] as a special case.

4. The Case of $l = \infty$

In this section we examine the convergence of the net $\{x_{s,t_s}\}_{s\in(0,1)}$ along the curve where $t_s/s \to \infty$, as $s \to 0$. We shall prove that the net converges strongly to a point $x_\infty \in S$ which is the unique solution of the VI (b) in Section 1.

Theorem 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$, $V, T : C \to C$ are nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$, and $f : C \to H$ is a ρ-contraction with coefficient $\rho \in [0, 1)$. Assume there holds the condition (A2) in Section 1. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \le \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $t_s = t(s)$ satisfy $\lim_{s\to 0} t_s/s = \infty$. Then the net $\{x_{s,t_s}\}_{s\in(0,1)}$ defined by

$$x_{s,t_s} = P_C[s\gamma f(x_{s,t_s}) + (I - s\mu F)W_{t_s}x_{s,t_s}], \tag{4.1}$$

where $W_{t_s} = t_sV + (1 - t_s)T$, strongly converges to $x_\infty \in S$ which is the unique solution of the VI (b).

Proof. The proof is divided into three steps, the first of which is to prove the boundedness of $\{x_{s,t_s}\}_{s\in(0,1)}$. Indeed, let $z \in S$. By condition (A2) there exists $p_s \in \mathrm{Fix}(W_s)$ such that $p_s \to z$ as $s \to 0$. Now, set

$$y_{s,t_s} = s\gamma f(x_{s,t_s}) + (I - s\mu F)W_{t_s}x_{s,t_s},$$

where $W_{t_s} = t_sV + (1 - t_s)T$. Then from (4.1) we obtain that $x_{s,t_s} = P_C y_{s,t_s}$ and

$$\begin{aligned} y_{s,t_s} - p_{t_s} &= s\gamma f(x_{s,t_s}) + (I - s\mu F)W_{t_s}x_{s,t_s} - p_{t_s} \\ &= s\gamma(f(x_{s,t_s}) - f(p_{t_s})) + s(\gamma f - \mu F)p_{t_s} + (I - s\mu F)W_{t_s}x_{s,t_s} - (I - s\mu F)W_{t_s}p_{t_s}. \end{aligned}$$

Since P_C is the metric projection from H onto C, utilizing Lemma 2.1, we have

$$\langle P_C y_{s,t_s} - y_{s,t_s}, P_C y_{s,t_s} - p_{t_s}\rangle \leq 0.$$

Thus utilizing Lemma 2.5 we get

$$\begin{aligned}
\|x_{s,t_s} - p_{t_s}\|^2 &= \langle P_C y_{s,t_s} - y_{s,t_s}, P_C y_{s,t_s} - p_{t_s}\rangle + \langle y_{s,t_s} - p_{t_s}, x_{s,t_s} - p_{t_s}\rangle \\
&\leq \langle y_{s,t_s} - p_{t_s}, x_{s,t_s} - p_{t_s}\rangle \\
&= s\gamma\langle f(x_{s,t_s}) - f(p_{t_s}), x_{s,t_s} - p_{t_s}\rangle + s\langle(\gamma f - \mu F)p_{t_s}, x_{s,t_s} - p_{t_s}\rangle \\
&\quad + \langle (I - s\mu F)W_{t_s}x_{s,t_s} - (I - s\mu F)W_{t_s}p_{t_s}, x_{s,t_s} - p_{t_s}\rangle \\
&\leq s\gamma\|f(x_{s,t_s}) - f(p_{t_s})\|\|x_{s,t_s} - p_{t_s}\| + s\langle(\gamma f - \mu F)p_{t_s}, x_{s,t_s} - p_{t_s}\rangle \\
&\quad + \|(I - s\mu F)W_{t_s}x_{s,t_s} - (I - s\mu F)W_{t_s}p_{t_s}\|\|x_{s,t_s} - p_{t_s}\| \\
&\leq s\gamma\rho\|x_{s,t_s} - p_{t_s}\|^2 + s\langle(\gamma f - \mu F)p_{t_s}, x_{s,t_s} - p_{t_s}\rangle + (1 - s\tau)\|x_{s,t_s} - p_{t_s}\|^2 \\
&= (1 - s(\tau - \gamma\rho))\|x_{s,t_s} - p_{t_s}\|^2 + s\langle(\gamma f - \mu F)p_{t_s}, x_{s,t_s} - p_{t_s}\rangle,
\end{aligned}$$

It follows that

$$\|x_{s,t_s} - p_{t_s}\|^2 \leq \frac{1}{\tau - \gamma\rho}\langle(\gamma f - \mu F)p_{t_s}, x_{s,t_s} - p_{t_s}\rangle. \tag{4.2}$$

This implies immediately that

$$\|x_{s,t_s} - p_{t_s}\| \leq \frac{1}{\tau - \gamma\rho}\|(\gamma f - \mu F)p_{t_s}\|. \tag{4.3}$$

From (4.3) the boundedness of $\{x_{s,t_s}\}_{s\in(0,1)}$ follows since $\{p_s\}$ is bounded.

The second step is to prove that the set of weak cluster points of $\{x_{s,t_s}\}_{s\in(0,1)}$, $\omega_w(x_{s,t_s})$, is a subset of S; moreover, $\omega_w(x_{s,t_s}) = \omega_S(x_{s,t_s})$. First observe that the boundedness of $\{x_{s,t_s}\}$, (3.8), and Lemma 2.2 imply that $\omega_w(x_{s,t_s}) \subset \mathrm{Fix}(T)$.

Now let $w \in \omega_w(x_{s,t_s})$ and assume that $x_n := x_{s_n,t_{s_n}} \rightharpoonup w$, where $s_n \to 0$. For convenience, we write $t_n = t_{s_n}$ for all n; thus, $t_n/s_n \to \infty$ as $n \to \infty$. Now, take a fixed $\widehat{x} \in \mathrm{Fix}(T)$ arbitrarily and set

$$y_n = s_n\gamma f(x_n) + (I - s_n\mu F)W_{t_n}x_n,$$

where $W_{t_n} = t_nV + (1 - t_n)T$. Then from (4.1) we obtain that $x_n = P_C y_n$ and

$$\begin{aligned}
y_n - \widehat{x} &= s_n\gamma(f(x_n) - f(\widehat{x})) + s(\gamma f(\widehat{x}) - \mu F W_{t_n}\widehat{x}) + (I - s_n\mu F)W_{t_n}x_n - (I - s_n\mu F)W_{t_n}\widehat{x} \\
&\quad + t_n(V - I)\widehat{x}.
\end{aligned}$$

Since P_C is the metric projection from H onto C, utilizing Lemma 2.1, we have

$$\langle P_C y_n - y_n, P_C y_n - \widehat{x}\rangle \leq 0.$$

Thus utilizing Lemma 2.5 we obtain that for a constant $M \geq \sup_n\{\|(\gamma f - \mu F W_{t_n})\widehat{x}\|\|x_n - \widehat{x}\|\}$,

$$\begin{aligned}
\|x_n - \widehat{x}\|^2 &= \langle P_C y_n - y_n, P_C y_n - \widehat{x}\rangle + \langle y_n - \widehat{x}, x_n - \widehat{x}\rangle \\
&\leq \langle y_n - \widehat{x}, x_n - \widehat{x}\rangle \\
&= s_n\gamma\langle f(x_n) - f(\widehat{x}), x_n - \widehat{x}\rangle + s_n\langle(\gamma f - \mu F W_{t_n})\widehat{x}, x_n - \widehat{x}\rangle \\
&\quad + \langle (I - s_n\mu F)W_{t_n}x_n - (I - s_n\mu F)W_{t_n}\widehat{x}, x_n - \widehat{x}\rangle + t_n\langle (V - I)\widehat{x}, x_n - \widehat{x}\rangle \\
&\leq s_n\gamma\|f(x_n) - f(\widehat{x})\|\|x_n - \widehat{x}\| + s_n\|(\gamma f - \mu F W_{t_n})\widehat{x}\|\|x_n - \widehat{x}\| \\
&\quad + \|(I - s_n\mu F)W_{t_n}x_n - (I - s_n\mu F)W_{t_n}\widehat{x}\|\|x_n - \widehat{x}\| + t_n\langle (V - I)\widehat{x}, x_n - \widehat{x}\rangle \\
&\leq s_n\gamma\rho\|x_n - \widehat{x}\|^2 + s_n M + (1 - s_n\tau)\|x_n - \widehat{x}\|^2 + t_n\langle (V - I)\widehat{x}, x_n - \widehat{x}\rangle \\
&= (1 - s_n(\tau - \gamma\rho))\|x_n - \widehat{x}\|^2 + t_n\langle (V - I)\widehat{x}, x_n - \widehat{x}\rangle + s_n M.
\end{aligned}$$

It follows that

$$\langle (I - V)\widehat{x}, x_n - \widehat{x}\rangle \leq \frac{s_n M}{t_n} \to 0$$

as $s_n/t_n \to 0$. But $x_n \rightharpoonup w$, and we derive

$$\langle (I - V)\widehat{x}, w - \widehat{x}\rangle \leq 0, \quad \forall \widehat{x} \in \mathrm{Fix}(T). \tag{4.4}$$

Upon replacing the $\widehat{x}$ in (4.4) with $w + \alpha(\widetilde{x} - w) \in \mathrm{Fix}(T)$, where $\alpha \in (0, 1)$ and $\widetilde{x} \in \mathrm{Fix}(T)$, we get

$$\langle (I - V)(w + \alpha(\widetilde{x} - w)), w - \widetilde{x}\rangle \leq 0.$$

Letting $\alpha \to 0$, we obtain the VI

$$\langle (I - V)w, w - \widetilde{x}\rangle \leq 0 \quad \forall \widetilde{x} \in \mathrm{Fix}(T).$$

Therefore, $w \in S$.

Next using condition (A2) again, we have a sequence $p_{t_n} \in \mathrm{Fix}(W_{t_n})$ such that $p_{t_n} \to w$. Then in relation (4.2) we replace z and p_{t_s} with w and p_{t_n}, respectively, to derive

$$\|x_n - p_{t_n}\|^2 \leq \frac{1}{\tau - \gamma\rho}\langle (\gamma f - \mu F)p_{t_n}, x_n - p_{t_n}\rangle. \tag{4.5}$$

Now since $(\gamma f - \mu F)p_{t_n} \to (\gamma f - \mu F)w$ and $x_n - p_{t_n} \rightharpoonup 0$, taking the limit in (4.5), we immediately get $x_n \to w$. Hence $w \in \omega_S(x_{s,t_s})$.

The third and final step is to prove that the net $\{x_{s,t_s}\}$ converges in norm to $x_\infty = P_S(I - \mu f + \gamma f)x_\infty$. It suffices to prove that each norm limit point $w \in \omega_S(x_{s,t_s})$ solves the VI (b) in Section 1. We still use the same subsequence $\{x_n\}$ of the net $\{x_{s,t_s}\}$ such that $x_n \to w$ as shown in the second step. On the other hand, for every $\bar{p} \in S$, by condition (A2), we have, for each n, $\bar{p}_{t_n} \in \mathrm{Fix}(W_{t_n})$ such that $\bar{p}_{t_n} \to \bar{p}$ as $n \to \infty$. Observe that

$$(\mu F - \gamma f)x_n = \frac{1}{s_n}(P_C y_n - y_n) - \frac{1}{s_n}(I - W_{t_n})x_n + \mu(F x_n - F W_{t_n} x_n),$$

where $y_n = s_n\gamma f(x_n) + (I - s_n\mu F)W_{t_n}x_n$ and $x_n = P_C y_n$. Utilizing Lemmas 2.1 and 2.5 we deduce from the monotonicity of $I - W_{t_n}$ that

$$\begin{aligned}
&\langle(\mu F - \gamma f)x_n, x_n - \bar{p}_{t_n}\rangle \\
&= \tfrac{1}{s_n}\langle P_C y_n - y_n, P_C y_n - \bar{p}_{t_n}\rangle - \tfrac{1}{s_n}\langle(I - W_{t_n})x_n, x_n - \bar{p}_{t_n}\rangle + \mu\langle Fx_n - FW_{t_n}x_n, x_n - \bar{p}_{t_n}\rangle \\
&= \tfrac{1}{s_n}\langle P_C y_n - y_n, P_C y_n - \bar{p}_{t_n}\rangle - \tfrac{1}{s_n}\langle(I - W_{t_n})x_n - (I - W_{t_n})\bar{p}_{t_n}, x_n - \bar{p}_{t_n}\rangle \\
&\quad + \mu\langle Fx_n - FW_{t_n}x_n, x_n - \bar{p}_{t_n}\rangle \\
&\le \mu\langle Fx_n - FW_{t_n}x_n, x_n - \bar{p}_{t_n}\rangle.
\end{aligned}$$

Note that $Fx_n - FW_{t_n}x_n = Fx_n - F[t_nVx_n + (1 - t_n)Tx_n] \to Fw - Fw = 0$ as $n \to \infty$. Passing to the limit as $n \to \infty$ in the last inequality, we conclude that

$$\langle(\mu F - \gamma f)w, w - \bar{p}\rangle \le 0, \quad \forall \bar{p} \in S.$$

This implies that w satisfies the VI (b) in Section 1. Hence $w = x_\infty$, as required. □

Remark 4.1. Compared with Theorem 3.1 of Cianciaruso, Colao, Muglia and Xu [13], our Theorem 4.1 improves and extends their Theorem 3.1 [13] in the following aspects:

(i) The (self) contraction $f : C \to C$ in [13, Theorem 3.1] is extended to the case of (possibly nonself) contraction $f : C \to H$ on a nonempty closed convex subset C of H.

(ii) The convex combination of (self) contraction f and nonexpansive mapping W_{t_s} in the implicit scheme (3.1) of Theorem 3.1 [13] is extended to the linear combination of (possibly nonself) contraction f and hybrid steepest-descent method involving W_{t_s}.

(iii) In order to guarantee that the net $\{x_{s,t_s}\}$ generated by the implicit scheme still lies in C, the implicit scheme (3.1) in [13, Theorem 3.1] is extended to develop our new implicit scheme (4.1) by virtue of the projection method.

(iv) The new technique of argument is applied to deriving our Theorem 4.1. For instance, the characteristic properties (Lemma 2.4) of the metric projection play a key role in proving the strong convergence of the net $\{x_{s,t_s}\}_{s\in(0,1)}$ in our Theorem 4.1.

(v) If we put $\mu = 1$, $F = I$ and $\gamma = \tau = 1$ and let f be a contractive self-mapping on C with coefficient $\rho \in [0, 1)$, then our Theorem 4.1 reduces to Theorem 3.1 [13]. Thus, our Theorem 4.1 covers Theorem 3.1 [13] as a special case.

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