A RECURRENCE METHOD FOR SIMPLE CONTINUOUS LINEAR PROGRAMMING PROBLEMS

Ching-Feng Wen*
General Education Center
Kaohsiung Medical University
100, Shih-Chuan 1st Road, Kaohsiung, Taiwan, 807, R.O.C.
Yan-Kuen Wu and Yung-Yih Lur
Department of Industrial Management
Vanung University

ABSTRACT

In this paper, we discuss a special class of continuous linear programming problems which can be called simple continuous linear programming problems (*SP*). A practical and efficient method for finding an approximate optimal value and optimal solution of (*SP*) is presented. The main work of computing an approximate optimal value in the provided method is only to solve finite linear programming problems by using recurrence relations. Furthermore, a simple algorithm can be employed not only to easily solve the (*SP*) problem but also to provide an error bound of optimal value as well. Some numerical examples are given to implement the algorithm.

***Keywords**:* continuous linear programming problems, recurrence method

1. INTRODUCTION

Continuous linear programming problems (*CLP*) proposed by Bellman [7, 8] in connection with the so-called bottleneck problems in multistage linear production processes, which are formulated as follows:

$$(CLP): \text{maximize } \int_0^T f(t)x(t)dt$$
$$\text{subject to } B(t)x(t)+\int_0^t K(s,t)x(s)ds \le g(t),$$
$$x(t)\ge 0,\ t\in[0,T],$$

where $B(t)$, $K(s, t)$ are given $m\times n$ matrices, $f(t)$ is a given n-vector, $g(t)$ is a given m-vector and $x(t)$ is an n-vector to be determined. Here all vectors are column vectors. In the literature, much research has been proposed to consider continuous linear programming problems. Studies on investigating a solution algorithm for (*CLP*), [9, 10, 14, 16, 24] provided a generation of the simplex method to a function space setting. Considering the duality of (*CLP*), [12, 13, 17, 25, 26] established strong duality theorems. Also, Ho-Lur-Wu [15] studied extreme points of the feasible region for a special class of continuous linear programming problem. Studying a special case of (*CLP*), Anderson [1] introduced the separated continuous linear programs (*SCLP*) to model job-shop scheduling problems. Since then many researches concerned with (*CLP*) has focused on (*SCLP*) [2, 4-6, 11, 18-23, 27]. One of a practical example of bottleneck problems is described as follows [3].

In an economy, n different goods, $G_1, G_2, \ldots, G_n$, are produced by m different types of plan or production facility, $P_1, P_2, \ldots, P_m$. At the beginning of a five-year plan, there is available a certain capacity in each of these types of plant, and more can be made by reinvesting the goods produced. The aim of the plan is to maximize the productive capacity at the end of the period.

Let $x_j(t)$, $j = 1, 2, \ldots, m$, denote the rate of production of new capacity of type j at time t. Production of new plant requires the consumption of a certain quantity b_{ij} of good G_i for each additional unit of plan P_j. Thus the amounts of goods consumed in this way are given by $Bx(t)$, where $x(t)$ is the vector with components $x_1(t), x_2(t), \ldots, x_m(t)$ and B is the matrix whose i, jth element is b_{ij}. Let $z_j(t)$ denote the total productive capacity of type j available at time t. Denote by d_{ij} the rate of production of G_i for each unit of plant P_j. Then the total rates of production of goods at time t are given by $Dz(t)$, where D is the matrix whose i, jth element is d_{ij} and $z(t)$ is the vector $(z_1(t), z_2(t), \ldots, z_m(t))^T$.

The constraint on investment in additional plant due to limitations in productive capacity is then given by $Bx(t)\le Dz(t)$ throughout the time period under consideration. If c_0 is the vector of initial productive capacities, we can write

$z(t)=c_0+\int_0^t x(s)ds \le c$, and hence

*Corresponding author: cfwen@cc.kmu.edu.tw

$Bx(t) - \int_0^t Dx(s)ds \le c$, where $c = Dc_0$. If we wish to maximize a weighted sum, $\sum a_i z_i(T)$, of the production capacities at the end of the time period, then we obtain the following linear program

$$\text{maximize} \quad \int_0^T a^T x(t)dt$$
$$\text{subject to} \quad Bx(t) - \int_0^t Dx(s)ds \le c,$$
$$x(t) \ge 0,\ t \in [0,T].$$

In this paper we discuss a special case of continuous linear programming problems. This continuous linear programming problem is defined as follows:

$$(SP)\text{: maximize} \quad \int_0^T f(t)x(t)dt$$
$$\text{subject to} \quad x(t) - \int_0^t x(s)ds \le g(t),\ \forall t \in [0,T]$$
$$x(t) \in L_\infty^+[0,T],$$

where f and g are continuous functions on $[0,T]$ and $L_\infty^+[0,T]$ is the set of nonnegative real valued, Lebesgue measurable, essentially bounded functions on $[0,T]$. The dual problem (*DSP*) of (*SP*) is defined as follows:

$$(DSP)\text{: maximize} \quad \int_0^T g(t)w(t)dt$$
$$\text{subject to} \quad w(t) - \int_t^T w(s)ds \ge f(t),\ \forall t \in [0,T]$$
$$w(t) \in L_\infty^+[0,T].$$

It is well known that (*SP*) and (*DSP*) have the weak duality property (see, for example [3]), that is, if $x(t)$ and $w(t)$ are feasible for (*SP*) and (*DSP*) respectively, then $\int_0^T f(t)x(t)dt \le \int_0^T g(t)w(t)dt$. Indeed, the (*SP*) problem we studied is a special case of Tyndall's work [25], which is an application to a dynamic closed end Leontief production model. By Theorem 1 of [25] there exist optimal solutions $x(t)$ and $w(t)$ in (*SP*) and (*DSP*) respectively such that $\int_0^T f(t)x(t)dt = \int_0^T g(t)w(t)dt$. However, Tyndall's work only verified the theoretical result of optimal solution. It seems too complex to find the optimal solution in the work of computation. Motivated by this factor, we intend to present an efficient algorithm to approach the optimal value of (*SP*) by using recurrence method. This method can be employed not only to easily solve the (*SP*) problem but also to provide an error bound of optimal value as well. For improving the readability, we define the notations $F(P)$ and $V(P)$ to be the feasible region and the optimal value of a linear programming problem (P), respectively. For example, $F(SP)$ is the feasible region and $V(SP)$ is the optimal value of (*SP*).

This paper is organized as follows. In Section 2, a discretization method is prepared for the proof of the main theorem in the following sections. Applying this discretization method we can obtain a sequence of optimal solutions of corresponding finite linear programming problems in Section 3. Then the corresponding value obtained from this sequence finally converges to the optimal value of (*SP*). In Section 4, we establish an error bound for the provided method and give some examples to illustrate the convergence of problem. Brief conclusion is given in Section 5.

2. PRELIMINARY RESULTS

For solving the (*SP*) and (*DSP*) problems, we use a discretization method for two functions f and g. For each $n \in N$, let $P_{2^n} = \{0, \frac{1}{2^n}T, \frac{2}{2^n}T, \ldots, \frac{2^n-1}{2^n}T, T\}$ be a partition on $[0,T]$. For $i = 1, 2, \ldots, 2^n$, let

$$b_i^{(n)} = \min\{g(x) : x \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T]\} \tag{1}$$

and

$$c_i^{(n)} = \min\{f(x) : x \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T]\}. \tag{2}$$

Step functions $f_n(t)$ and $g_n(t)$ are defined as follows:

$$f_n(t) = \begin{cases} c_i^{(n)}, & \text{for } t \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T), \\ c_{2^n}^{(n)}, & \text{for } t = T, \end{cases} \tag{3}$$

and

$$g_n(t) = \begin{cases} b_i^{(n)}, & \text{for } t \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T), \\ b_{2^n}^{(n)}, & \text{for } t = T. \end{cases} \tag{4}$$

Consider the following programming problem:

$$(SP_n)\text{ : maximize} \quad \int_0^T f_n(t)x(t)dt$$
$$\text{subject to} \quad x(t) - \int_0^t x(s)ds \le g_n(t), \forall t \in [0,T]$$
$$x(t) \in L_\infty^+[0,T],$$

and its dual problem:

$$(DSP_n)\text{: minimize} \quad \int_0^T g_n(t)w(t)dt$$
$$\text{subject to} \quad w(t) - \int_t^T w(s)ds \ge f_n(t),\ \forall t \in [0,T]$$
$$w(t) \in L_\infty^+[0,T].$$

Assumption 1. f and g are continuous on $[0,T]$, and $g(t) > 0$ for all $t \in [0, T]$.

In this article, we always employ Assumption 1 to solve the continuous linear programming problems

(*SP*) and (*DSP*). Note that, under Assumption 1, we have the following properties.

(1) (*SP*) and (SP_n) are feasible for all $n \in N$, since the zero function is a common feasible solution of (*SP*) and (SP_n). Moreover, let $w_*(t) \triangleq ke^{T-t}$, where $k = \max_{t\in[0,T]}\{f(t),\ 0\}$. Then $w_*(t) - \int_t^T w_*(s)ds = k \geq f(t) \geq f_n(t)$, for all $t \in [0,\ T]$. Hence $w_*(t)$ is a common feasible solution of (*DSP*) and (DSP_n) for all $n \in N$. Therefore, (*SP*), (*DSP*), (SP_n) and (DSP_n) are all feasible.

(2) Since $g_1(t) \leq g_2(t) \leq \cdots \leq g(t)$ and $f_1(t) \leq f_2(t) \leq \cdots \leq f(t)$ for all $t \in [0, T]$, we have

$F(SP_1) \subseteq F(SP_2) \subseteq \cdots \subseteq F(SP)$

and

$F(DSP_1) \supseteq F(DSP_2) \supseteq \cdots \supseteq F(DSP)$,

which implies

$$-\infty < V(SP_1) \leq V(SP_2) \leq \cdots \leq V(SP) < \infty \tag{5}$$

and

$-\infty < V(DSP_1) \leq V(DSP_2) \leq \cdots \leq V(DSP) < \infty$.

Hence,

$\lim_{n\to\infty} V(SP_n) \leq V(SP)$

and

$$\lim_{n\to\infty} (DSP_n) \leq V(DSP). \tag{6}$$

Lemma 1. Suppose that Assumption 1 holds. Let $w^{(n)}(t)$ be a feasible solution of (DSP_n), $\varepsilon = \sup_{t\in[0,T]}\{f(t) - f_n(t)\}$ and $\varepsilon' = \sup_{t\in[0,T]}\{g(t) - g_n(t)\}$. Let $\tilde{w}^{(n)}(t) = w^{(n)}(t) + \varepsilon e^{T-t}$. Then $\tilde{w}^{(n)}(t)$ is a feasible solution of (*DSP*) and

$$0 \leq \int_0^T g(t)\tilde{w}^{(n)}(t)dt - \int_0^T g_n(t)w^{(n)}(t)dt$$
$$\leq \varepsilon' \int_0^T w^{(n)}(t)dt + \varepsilon \int_0^T g(t)e^{T-t}dt.$$

Proof. Observe that $\tilde{w}^{(n)}(t) \geq w^{(n)}(t) \geq 0$ and

$$\begin{aligned}&\tilde{w}^{(n)}(t) - \int_t^T \tilde{w}^{(n)}(s)ds\\ &= w^{(n)}(t) + \varepsilon e^{T-t} - \int_t^T w^{(n)}(s)ds - \int_t^T \varepsilon e^{T-s}ds\\ &= w^{(n)}(t) - \int_t^T w^{(n)}(s)ds + \varepsilon\\ &\geq f_n(t) + \varepsilon\\ &\geq f(t) \text{ for all } t \in [0,T].\end{aligned}$$

Hence $\tilde{w}^{(n)}(t) \in F(DSP)$. Since $\tilde{w}^{(n)}(t) \geq w^{(n)}(t) \geq 0$ and $g(t) \geq g_n(t) \geq 0$, we have

$$\begin{aligned}0 &\leq \int_0^T g(t)\tilde{w}^{(n)}(t)dt - \int_0^T g_n(t)w^{(n)}(t)dt\\ &= \int_0^T [g(t) - g_n(t)]w^{(n)}(t)dt + \varepsilon \int_0^T g(t)e^{T-t}dt\\ &\leq \varepsilon' \int_0^T w^{(n)}(t)dt + \varepsilon \int_0^T g(t)e^{T-t}dt.\end{aligned}$$

Based on Lemma 1, the optimal value of (*DSP*) can be obtained as the following theorem.

Theorem 1. Under Assumption 1, we have $\lim_{n\to\infty} V(DSP_n) = V(DSP)$.

Proof. Let $\varepsilon > 0$ be given. By continuity of f and g, there exists a positive integer N such that for all $n \geq N$ $f_n(t) \leq f(t) \leq f_n(t) + \varepsilon$ and $g_n(t) \leq g(t) \leq g_n(t) + \varepsilon$, for all $t \in [0,T]$. Here $f_n(t)$ and $g_n(t)$ are defined as in (3) and (4). Let $w_0(t)$ be a feasible solution of (*DSP*) and $\int_0^T w_0(t)dt = \theta$. For this $\varepsilon > 0$, we claim that for all $n \geq N$ there exists $w^{(n)}(t) \in F(DSP_n)$ such that

$$0 \leq \int_0^T g_n(t)w^{(n)}(t)dt - V(DSP_n) \leq \varepsilon \tag{7}$$

and

$$\int_0^T w^{(n)}(t)dt \leq \frac{M}{m}\theta, \tag{8}$$

where $M = \max_{t\in[0,T]} g(t)$ and $m = \min_{t\in[0,T]} g(t)$. We prove this claim by distinguishing the following two cases.

Case 1. $\int_0^T g_n(t)w_0(t)dt - V(DSP_n) \leq \varepsilon$. Put $w^{(n)}(t) = w_0(t)$. Then (7) holds and

$\int_0^T w^{(n)}(t)dt = \int_0^T w_0(t)dt = \theta \leq \frac{M}{m}\theta.$

Case2. $\int_0^T g_n(t)w_0(t)dt - V(DSP_n) > \varepsilon.$ Then there exists $w^{(n)}(t) \in F(DSP_n)$ such that

$$V(DSP_n) \leq \int_0^T g_n(t)w^{(n)}(t)dt \leq \int_0^T g_n(t)w_0(t)dt \tag{9}$$

and

$\int_0^T g_n(t)w^{(n)}(t)dt - V(DSP_n) \leq \varepsilon.$

Suppose that $\int_0^T w^{(n)}(t)dt > \frac{M}{m}\theta.$ As $g_n(t) \geq m$ for all t, we have

$$\begin{aligned}&\int_0^T g_n(t)w^{(n)}(t)dt > m\frac{M}{m}\theta = M\theta\\ &= M\int_0^T w_0(t)dt \geq \int_0^T g_n(t)w_0(t)dt,\end{aligned}$$

which contradicts (9),

hence $\int_0^T w^{(n)}(t)dt \le \frac{M}{m}\theta$. Therefore, (7) and (8) hold.

Now for $n \ge N$ we define a function $\tilde{w}^{(n)}(t) = w^{(n)}(t) + \varepsilon e^{T-t}$ for all $t \in [0, T]$. By Lemma 1, $\tilde{w}^{(n)}$ is a feasible solution of (*DSP*) and

$$\begin{aligned} 0 &\le \int_0^T g(t)\tilde{w}^{(n)}(t)dt - \int_0^T g_n(t)w^{(n)}(t)dt \\ &\le \varepsilon[\int_0^T w^{(n)}(t)dt + \int_0^T g(t)e^{T-t}dt] \\ &\le \varepsilon[\frac{M}{m}\theta + \int_0^T g(t)e^{T-t}dt], \end{aligned} \tag{10}$$

by (8). Moreover, observe that

$$\begin{aligned} 0 &\le \int_0^T g(t)\tilde{w}^{(n)}(t)dt - V(DSP_n) \\ &= \int_0^T g(t)\tilde{w}^{(n)}(t)dt - \int_0^T g_n(t)w^{(n)}(t)dt \\ &+ \int_0^T g_n(t)w^{(n)}(t)dt - V(DSP_n) \\ &\le \varepsilon[\frac{M}{m}\theta + \int_0^T g(t)e^{T-t}dt] + \varepsilon, \end{aligned}$$

by (7) and (10). Since ε is arbitrary, $\lim_{n\to\infty}\int_0^T g(t)\tilde{w}^{(n)}(t)dt = \lim_{n\to\infty} V(VSP_n)$. Owing to the fact that $\int_0^T g(t)\tilde{w}^{(n)}(t)dt \ge V(DSP)$, we obtain $\lim_{n\to\infty} V(DSP_n) \ge V(DSP)$. Applying this result and (6), we have $\lim_{n\to\infty} V(DSP_n) = V(DSP)$. This completes the proof.

3. A RECURRENCE METHOD FOR (*SP*)

In this section we discuss the finite dimensional linear programming problem (P_n) which is due to (*SP*). A recurrence method is then provided for solving the dual problem of (P_n) and verifying the optimal value of (P_n) is just to the approximate optimal value of (*SP*).

For each $n \in N$, let $b_i^{(n)}$ and $c_i^{(n)}$ be defined as in (1) and (2). Note that under Assumption 1, we have $b_i^{(n)} > 0$ for all *i*. Now we define the following linear programming problem:

$$(P_n): \text{maximize } \sum_{i=1}^{2^n} \frac{Tc_i^{(n)}x_i}{2^n}$$

$$\text{subject to } \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ \frac{-T}{2^n} & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_2 \end{bmatrix} \le \begin{bmatrix} b_1^{(n)} \\ \vdots \\ b_{2^n}^{(n)} \end{bmatrix}$$

$$x_i \ge 0,\ \forall i = 1,\dots,2^n.$$

It is easy to see that the zero vector is a feasible solution of (P_n) and the dual problem (D_n) of (P_n) is defined as follows:

$$(D_n): \text{minimize } \sum_{i=1}^{2^n} \frac{Tb_i^{(n)}w_i}{2^n}$$

$$\text{subject to } \begin{bmatrix} 1 & & \frac{-T}{2^n} \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_{2^n} \end{bmatrix} \ge \begin{bmatrix} c_1^{(n)} \\ \vdots \\ c_{2^n}^{(n)} \end{bmatrix}$$

$$w_i \ge 0,\ \forall i = 1,\dots,2^n.$$

Remark 1. Under Assumption 1, it is easy to obtain an optimal solution of (D_n) via the following recurrence method; hence, through the strong duality theory of finite linear programming we have $-\infty < V(P_n) = V(D_n) < \infty$. To find an optimal solution of (D_n), let $\overline{w}^{(n)} = (\overline{w}_1^{(n)},\dots,\overline{w}_{2^n}^{(n)})^T$, where $\overline{w}_{2^n}^{(n)} = \max\{c_{2^n}^{(n)}, 0\}$ and

$$\overline{w}_1^{(n)} = \max\{c_i^{(n)} + \frac{T}{2^n}\sum_{l=i+1}^{2^n} \overline{w}^{(n)}, 0\},\ i = 1,2,\dots,2^n - 1.$$

It is obvious that $\overline{w}^{(n)}$ is a feasible solution of (D_n). Now we show that $\overline{w}^{(n)}$ is an optimal solution of (D_n). Let $w = (w_1, w_2, \dots, w_{2^n})^T \in F(D_n)$ be given. Clearly, $w_{2^n} \ge \max\{c_{2^n}^{(n)}, 0\} = \overline{w}_{2^n}^{(n)}$. We claim $w_k \ge \overline{w}_k^{(n)}$ for all $k = 1, 2, \dots 2^n$, and prove it by induction. Suppose that $w_j \ge \overline{w}_j^{(n)}$ for all $j = k+1, k+2, \dots, 2^n$. Since w is a feasible solution, we have $w_k \ge \frac{T}{2^n}(w_{k+1} + w_{k+2} + \dots + w_{2^n}) + c_k^{(n)}$. Thus,

$$\begin{aligned} w_k &\ge \max\{c_k^{(n)} + \frac{T}{2^n}(w_{k+1} + w_{k+2} + \dots + w_{2^n}), 0\} \\ &\ge \max\{c_k^{(n)} + \frac{T}{2^n}(\overline{w}_{k+1}^{(n)} + \overline{w}_{k+2}^{(n)} + \dots + \overline{w}_{2^n}^{(n)}), 0\} \\ &= \overline{w}_k^{(n)}. \end{aligned}$$

Therefore, $w_k \ge \overline{w}_k^{(n)}$ for all $k = 1, 2, \dots 2^n$. This proves the claim. Applying Assumption 1, $b_i^{(n)} > 0$ for all *i*; hence, $\sum_{i=1}^{2^n} \frac{T}{2^n} b_i^{(n)} w_i \ge \sum_{i=1}^{2^n} \frac{T}{2^n} b_i^{(n)} \overline{w}_i^{(n)}$. Since $w \in F(D_n)$ is arbitrary, we see that $\overline{w}^{(n)}$ is an optimal solution of (D_n).

After the discussion of optimal solution for (P_n) and (D_n) problems, the optimal values of relationship between $V(SP_n)$ and $V(P_n)$ are considered as follows.

Theorem 2. Suppose that Assumption 1 holds. Then $V(SP_n) \ge V(Pn)$, for all $n = 1, 2, \dots$.

Proof. Let $x = (x_1,\dots,x_{2^n})^T$ be an optimal solution of (P_n). Define $x(t)$ by

$$x(t) = \begin{cases} x_i, & \text{for } \frac{(i-1)T}{2^n} \le t < \frac{iT}{2^n}, i = 1,2,\dots,2^n \\ x_{2^n}, & \text{for } t = T. \end{cases}$$

Case 1. $t \in [\frac{(i-1)T}{2^n}, \frac{iT}{2^n})$, where $1 \le i \le 2^n$. Then

$$x(t)-\int_0^t x(s)ds$$
$$=x_i-\sum_{j=1}^{i-1}\frac{T}{2^n}x_j-\int_{\frac{(i-1)T}{2^n}}^{t}x(s)ds$$
$$\le x_i-\sum_{j=1}^{i-1}\frac{T}{2^n}x_j\le b_i^{(n)}\le g_n(t).$$

Case 2. $t = T$. Then

$$x(T)-\int_0^T x(s)ds=x_{2^n}-\sum_{i=1}^{2^n}\frac{T}{2^n}x_i$$
$$\le x_{2^n}-\sum_{i=1}^{2^n-1}\frac{T}{2^n}x_i\le b_{2n}^{(n)}\le g_n(T).$$

By case 1 and case 2, $x(t)$ is a feasible solution for (SP_n). It is easy to see that

$$\int_0^T f_n(t)x(t)dt=\sum_{i=1}^{2^n}\frac{T}{2^n}c_i^{(n)}x_i=V(P_n).$$

Therefore, $V(SP_n)\ge V(P_n)$. This completes the proof.

According to Theorem 2, one can easy to see that $V(DSP_n)\ge V(SP_n)\ge V(P_n)=V(D_n)$. That is

$V(DSP_n)\ge V(D_n)$, for all $n=1,2,\ldots$ (11)

By inequality (5) and Theorem 2, one can easily see that

$V(DSP)\ge V(SP)\ge V(SP_n)\ge V(P_n)=V(D_n)$,
for all $n=1,2,\ldots$. (12)

Theorem 3. Suppose that Assumption 1 holds. Then $\lim_{n\to\infty}V(D_n)=\lim_{n\to\infty}V(DSP_n)$.

Proof. Let $\overline{w}^{(n)}=(\overline{w}_1^{(n)},\ldots,\overline{w}_{2^n}^{(n)})^T$, as given in Remark 1, be an optimal solution of (D_n). Define a function $\hat{w}(t)$ by

$$\hat{w}=\begin{cases}\overline{w}_i^{(n)}+\delta_{2^n}e^{T-t}, & \text{for } \frac{i-1}{2^n}T\le t\le\frac{i}{2^n}T, i=1,2,\ldots 2^n\\ \overline{w}_{2^n}^{(n)}+\delta_{2^n}, & \text{for } t=T,\end{cases} \quad (13)$$

where $\delta_{2^n}=\max\{\frac{T}{2^n}\overline{w}_j^{(n)}: j=1,\ldots,2^n\}$. We first show that $\hat{w}(t)\in F(DSP_n)$. For $t\in[\frac{i-1}{2^n}T,\frac{i}{2^n}T)$, we have

$$\hat{w}(t)-\int_t^T\hat{w}(s)ds$$
$$=\hat{w}(t)-[\int_t^{\frac{i}{2^n}T}\hat{w}(s)ds+\sum_{j=i+1}^{2^n}\int_{\frac{j-1}{2^n}T}^{\frac{j}{2^n}T}\hat{w}(s)ds]$$
$$=(\overline{w}_i^{(n)}+\delta_{2^n}e^{T-t})-[\overline{w}_i^{(n)}(\tfrac{i}{2^n}T-t)+\int_t^{\frac{i}{2^n}T}\delta_{2^n}e^{T-s}ds+\sum_{j=i+1}^{2^n}\frac{T}{2^n}\overline{w}_j^{(n)}+\sum_{j=i+1}^{2^n}\int_{\frac{j-1}{2^n}T}^{\frac{j}{2^n}T}\delta_{2^n}e^{T-s}ds]$$
$$=\overline{w}_i^{(n)}+\delta_{2^n}e^{T-t}-\overline{w}_i^{(n)}(\tfrac{i}{2^n}T-t)-\sum_{j=i+1}^{2^n}\tfrac{T}{2^n}\overline{w}_j^{(n)}+\delta_{2^n}(1-e^{T-t})$$
$$=\overline{w}_i^{(n)}-\sum_{j=i+1}^{2^n}\frac{T}{2^n}\overline{w}_j^{(n)}-\overline{w}_i^{(n)}(\frac{i}{2^n}T-t)+\delta_{2^n}$$
$$\ge\overline{w}_i^{(n)}-\sum_{j=i+1}^{2^n}\tfrac{T}{2^n}\overline{w}_j^{(n)}-\frac{T}{2^n}\overline{w}_i^{(n)}+\delta_{2^n}$$
$$\ge\overline{w}_i^{(n)}-\sum_{j=i+1}^{2^n}\tfrac{T}{2^n}\overline{w}_j^{(n)}\quad(\text{Since }\delta_{2^n}\ge\frac{T}{2^n}\overline{w}_i^{(n)})$$
$$\ge c_i^{(n)}=f_n(t).$$

Also, it is clear that $\hat{w}(T)-\int_T^T\hat{w}(s)ds=\hat{w}_{2^n}^{(n)}+\delta_{2^n}\ge c_{2^n}^{(n)}=f_n(T)$, hence $\hat{w}(t)$ is a feasible solution of (DSP_n). Moreover, observe that

$$\int_0^T g_n(t)\hat{w}(t)dt=\sum_{i=1}^{2^n}b_i^{(n)}\frac{T}{2^n}\overline{w}_i^{(n)}+\delta_{2^n}\int_0^T g_n(t)e^{T-t}dt$$
$$\le\sum_{i=1}^{2^n}b_i^{(n)}\frac{T}{2^n}\overline{w}_i^{(n)}+\delta_{2^n}\int_0^T g(t)e^{T-t}dt$$
$$=V(D_n)+\delta_{2^n}\int_0^T g(t)e^{T-t}dt.$$

This and (11) together imply that

$$V(D_n)\le V(DSP_n)\le V(D_n)+\delta_{2^n}\int_0^T g(t)e^{T-t}dt. \quad (14)$$

We now claim that $\delta_{2^n}\to 0$ as $n\to\infty$. To prove this claim, we need to prove the following fact by induction:

$\overline{w}_{2^n-j}^{(n)}\le(1+\frac{T}{2^n})^jL$, for all $j=0,1,2,\ldots,2^n-1$,

where $L=\max_{0\le t\le T}\{f(t),\ 0\}$. Note that $\max_{1\le i\le 2^n}\{c_i^{(n)},\ 0\}\le L$. It is obvious that $\overline{w}_{2^n}^{(n)}=\max\{c_{2^n}^{(n)},\ 0\}\le L$ and

$$\overline{w}_{2^n-1}^{(n)}=\max\{c_{2^n-1}^{(n)}+\frac{T}{2^n}\overline{w}_{2^n}^{(n)},\ 0\}$$
$$\le\max\{c_{2^n-1}^{(n)},\ 0\}+\frac{T}{2^n}\overline{w}_{2^n}^{(n)}\le L+\frac{T}{2^n}L=(1+\frac{T}{2^n})L.$$

Suppose that

$\overline{w}_{2^n-k}^{(n)}\le(1+\frac{T}{2^n})^kL$, for all $k=0,1,2,\ldots,j-1$.

Then

$$\overline{w}^{(n)}_{2^n-j} = \max\{c^{(n)}_{2^n-1} + \frac{T}{2^n}\sum_{k=0}^{j-1}\overline{w}^{(n)}_{2^n-k},\ 0\}$$
$$\le \max\{c^{(n)}_{2^n-1},\ 0\} + \frac{T}{2^n}\sum_{k=0}^{j-1}\overline{w}^{(n)}_{2^n-k}$$
$$\le L + \frac{T}{2^n}\sum_{k=0}^{j-1}(1+\frac{T}{2^n})^k L$$
$$= (1+\frac{T}{2^n})^j L.$$

This proves the fact, which implies

$$\overline{w}^{(n)}_j \le (1+\frac{T}{2^n})^{2^n-j} L \le (1+\frac{T}{2^n})^{2^n} L \le e^T L,$$

for all $j = 0,1,2,\ldots,2^n-1$.

Thus,

$$0 \le \delta_{2^n} = \max_{1\le j\le 2^n}\{\frac{T}{2^n}\overline{w}^{(n)}_j\} \le \frac{T}{2^n}e^T L.$$

Hence,

$$\delta_{2^n} \to 0 \text{ as } n\to\infty.$$

This proves the claim. Applying (14), (15), and Theorem 2, we see that $\lim_{n\to\infty} V(D_n) = \lim_{n\to\infty} V(DSP_n)$. This completes the proof.

By inequality (12), Theorem 3 and Theorem 1, we have $V(DSP) \ge V(SP) \ge \lim_{n\to\infty} V(D_n) = V(DSP)$. Therefore, $V(DSP) = V(SP) = \lim_{n\to\infty} V(D_n)$, so we have the following theorem.

Theorem 4. Suppose that Assumption 1 holds. Then $V(DSP)=V(SP)=\lim_{n\to\infty} V(D_n)$; that is, there is no duality gap between (SP) and (DSP).

4. THE SOLUTION ALGORITHM

In this section, we discuss an error bound between $V(DSP)$ and $V(D_n)$. An algorithm to approach the optimal value of (SP) is established as well. Let $\overline{w}^{(n)} = (\overline{w}^{(n)}_1,\ldots,\overline{w}^{(n)}_{2^n})^T$ be an optimal solution of (D_n) as given in Remark 1. Let $\delta_{2^n} = \max\{\frac{T}{2^n}\overline{w}^{(n)}_j : j = 1,\ldots 2^n\}$, $\varepsilon = \sup_{t\in[0,T]}\{f(t) - f_n(t)\}$, and $\varepsilon' = \sup_{t\in[0,T]}\{g(t) - g_n(t)\}$. Then we have the following result.

Theorem 5. Suppose that Assumption 1 holds. Then $0 \le V(DSP) - V(D_n) \le \varepsilon_n$, where

$$\varepsilon_n = \varepsilon'[\sum_{i=1}^{2^n}\tfrac{T}{2^n}\overline{w}^{(n)}_i + \delta_{2^n}(e^T-1)] + (\varepsilon+\delta_{2^n})\int_0^T g(t)e^{T-t}dt.$$

Proof. Let $\overline{w}^{(n)} = (\overline{w}^{(n)}_1,\ldots,\overline{w}^{(n)}_{2^n})^T$ be an optimal solution of (D_n) and $\hat{w}^{(n)}(t)$ be defined as in (13). Through the proof of Theorem 3, we see that $\hat{w}^{(n)}(t)$ is a feasible solution for (DSP_n), and $\int_0^T g_n(t)\hat{w}^{(n)}(t)dt \le V(D_n) + \delta_{2^n}\int_0^T g(t)e^{T-t}dt$. Let $\tilde{w}^{(n)}(t) = \hat{w}^{(n)}(t) + \varepsilon e^{T-t}$. By Lemma 1, $\tilde{w}^{(n)}(t)$ is a feasible solution of (DSP) and

$$0 \le \int_0^T g(t)\tilde{w}^{(n)}(t)dt - \int_0^T g_n(t)\hat{w}^{(n)}(t)dt$$
$$\le \varepsilon'\int_0^T \hat{w}^{(n)}(t)dt + \varepsilon\int_0^T g(t)e^{T-t}dt$$
$$= \varepsilon'(\sum_{i=1}^{2^n}\int_{\frac{i-1}{2^n}T}^{\frac{i}{2^n}T}\overline{w}^{(n)}_i dt + \int_0^T \delta_{2^n}e^{T-t}dt) + \varepsilon\int_0^T g(t)e^{T-t}dt$$
$$= \varepsilon'[\sum_{i=1}^{2^n}\frac{T}{2^n}\overline{w}^{(n)}_i + \delta_{2^n}(e^T-1)] + \varepsilon\int_0^T g(t)e^{T-t}dt.$$

Thus,

$$0 \le V(DSP) - V(D_n)$$
$$\le \int_0^T g(t)\tilde{w}^{(n)}(t)dt - V(D_n)$$
$$= \int_0^T g(t)\tilde{w}^{(n)}(t)dt - \int_0^T g_n(t)\hat{w}^{(n)}(t)dt$$
$$+ \int_0^T g_n(t)\hat{w}^{(n)}(t)dt - V(D_n)$$
$$\le \varepsilon'[\sum_{i=1}^{2^n}\frac{T}{2^n}\overline{w}^{(n)}_i + \delta_{2^n}(e^T-1)] + (\varepsilon+\delta_{2^n})\int_0^T g(t)e^{T-t}dt.$$

This completes the proof.

According to Theorems 4 and 5, we summarize the preceding discussions to form the following procedure for finding the approximate optimal value of (SP).

Algorithm:

Let δ be a prescribed small positive number, and an initial number $n_0 \in N$ be given.

Step 1. Set $n \leftarrow n_0$.

Step 2. Calculate $\overline{w}^{(n)}_i$ for $i = 1,2,\ldots,2^n$ by the recurrence method as in Remark 1. Evaluate the error bound ε_n as defined in Theorem 5.

Step 3. If $\varepsilon_n \le \delta$, then stop and evaluate the value $\sum_{i=1}^{2^n}\frac{T}{2^n}b^{(n)}_i\overline{w}^{(n)}_i$ as the approximate value of this problem. Otherwise, update $n \leftarrow n + 1$ and go to Step 2.

Remark 2. From Theorem 4, we can approach the value $V(SP)$ by $V(D_n)$. Note that by the complementary slackness theorem of finite linear programming, we can via the optimal solution $\overline{w}^{(n)}$ of (D_n) to obtain an optimal solution of (P_n), say $\overline{x}^{(n)} = (\overline{x}^{(n)}_1,\ldots,\overline{x}^{(n)}_{2^n})^T$. Define $x^{(n)}(t)$ by

$$x^{(n)}(t) = \begin{cases} \overline{x}^{(n)}_i, & \text{if } t\in[\frac{i-1}{2^n}T, \frac{i}{2^n}T),\ i = 1,2,\ldots,2^n \\ \overline{x}^{(n)}_{2^n}, & \text{if } t = T. \end{cases}$$

Applying the same argument as the proof of Theorem 2, we see that $x^{(n)}(t) \in F(SP_n)$ (hence $x^{(n)}(t) \in F(SP)$) and $\int_0^T f_n(t)x^{(n)}(t)dt = V(P_n)$. Since $f_n(t) \le f(t)$, by Theorem 5, we have

$$0 \le V(SP) - \int_0^T f(t)x^{(n)}(t)dt$$
$$\le V(SP) - \int_0^T f_n(t)x^{(n)}(t)dt \le \varepsilon_n,$$

where ε_n is defined as in Theorem 5. Therefore, the value $\int_0^T f(t)x^{(n)}(t)dt$ is an approximate value of (*SP*), and the error between the optimal value and the approximate value is less or equal to ε_n.

For illustration purpose, we use two examples to show that the proposed scheme works for real.

Example 1.

maximize $\int_0^1 (t^3 - 4t + 1)x(t)dt$

subject to $x(t) - \int_0^t x(s)ds \le t + 1, \forall t \in [0,1]$

$x(t) \in L_\infty^+[0,1].$

Example 2.

maximize $\int_0^1 t^2 \sin(7t)x(t)dt$

subject to $x(t) - \int_0^t x(s)ds \le 2 + \cos(5t), \forall t \in [0,1]$

$x(t) \in L_\infty^+[0,1].$

To illustrate the convergence, we select the partition number n from 1 to 20 and put $c_i^{(n)} = \min\{f(x): x \in [\frac{i-1}{2^n}, \frac{i}{2^n}]\}$ and $b_i^{(n)} = \min\{g(x): x \in [\frac{i-1}{2^n}, \frac{i}{2^n}]\}$, for all $i = 1, 2, \ldots, 2^n$. Using MATLAB Version 7.0.1 on a PC for the experiment, the results obtained by running the program which implement the proposed algorithm are presented in Table 1.

Table 1. Approximate value and error bound for Examples

Example 1		Example 2	
Approximate value	Error bound	Approximate value	Error bound
0	4.5685569	0	3.8425631
0.0039063	2.4053556	0	3.8146648
0.0651855	1.3453512	0.0223334	2.6425267
0.1037215	0.7187507	0.0813532	1.4580851
0.1253131	0.3725119	0.1227353	0.7575707
0.1367418	0.1897665	0.1471074	0.3857337
0.1426215	0.0957909	0.1607905	0.1942189
0.1456082	0.0481262	0.1678167	0.0974643
0.1471327	0.0241213	0.1713737	0.0488146
0.1478977	0.0120753	0.1731669	0.0244271
0.1482809	0.0060413	0.1740682	0.0122184
0.1484726	0.0030216	0.1745198	0.0061104
0.1485686	0.0015110	0.1747459	0.0030555
0.1486166	0.0007556	0.1748590	0.0015278
0.1486406	0.0003778	0.1749156	0.0007639
0.1486526	0.0001889	0.1749439	0.0003820
0.1486586	0.0000945	0.1749580	0.0001910
0.1486616	0.0000472	0.1749651	0.0000955
0.1486631	0.0000236	0.1749686	0.0000477
0.1486639	0.0000118	0.1749704	0.0000239

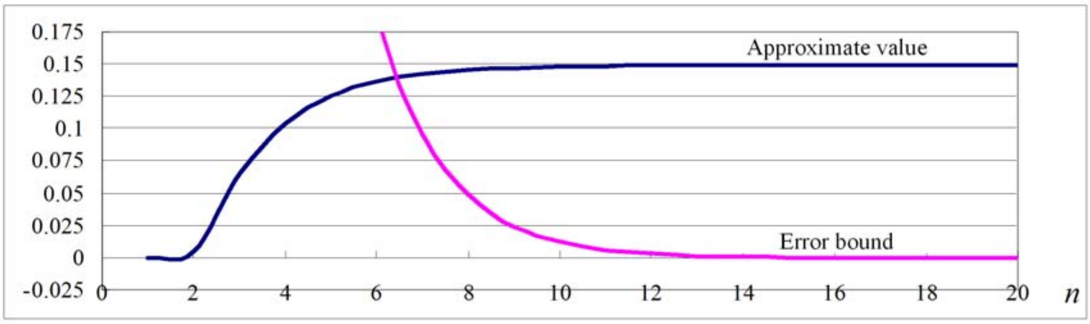

Figure 1. The trending diagram of Example 1

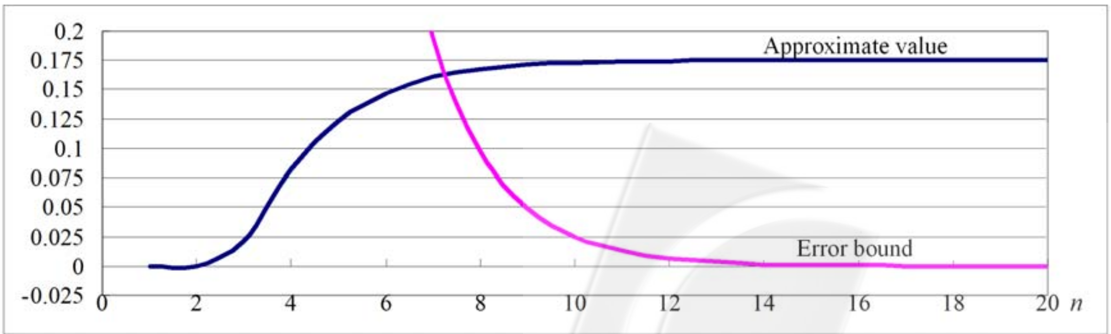

Figure 2. The trending diagram of Example 2

Depending on the approximate values and error bounds in Table 1, trending diagrams were constructed as represented in Figures 1 and 2 for Examples 1 and 2, respectively. From these figures, we can easily observe how these continuous linear programming problems come to a convergent value.

5. CONCLUSION

In the literature, some classic contributions introduced the separated concept to investigate the continuous linear programming problems. However, most of these researches focused on verifying the theoretical result of optimal solution. In this study, we concerned with how the approximate optimal value and optimal solution of simple continuous linear programming problems (*SP*) can be easily obtained. First using a discretization method we obtain finite dimensional linear programming problems (P_n) from (*SP*). Then a recurrence method is provided for solving the dual problem of (P_n) and we also verify that its optimal value is just to the approximate optimal value of (*SP*). Moreover, the optimal solution of (P_n) can be derived by the complementary slackness theorem. Based on the optimal solution of (P_n), we can easily construct an approximate optimal solution of the simple continuous linear programming problem (*SP*).

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ABOUT THE AUTHORS

Ching-Feng Wen is currently an assistant professor in General Education Center at Kaohsiung Medical University, Taiwan. He received his M.S. in Department of Mathematics from the National Kaohsiung Normal University, Taiwan and the Ph.D. degree in Department of Mathematics from National Cheng Kung University, Taiwan. His present research interests include linear programming, fractional programming and infinite-dimensional optimization.

Yan-Kuen Wu is a professor in the Department of Industrial Management at Vanung University, Taiwan. He received his M.S. degree in Industrial Management from the National Cheng Kung University, Tainan, Taiwan, in 1984 and the Ph.D. degree in Industrial Engineering from the Yuan Ze University, Taoyuan, Taiwan, in 1997. His research interests include fuzzy optimization and its applications, and mathematical programming.

Yung-Yih Lur is a professor in the Department of Industrial Management at Vanung University, Taiwan. He received his Ph.D. degree in Department of Mathematics from Chung Yuan Christian University, Taiwan, in 1999. His research interests include fuzzy matrices and its applications, and mathematical programming.

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以遞迴方法求解連續型的線性規劃問題

溫慶豐*
高雄醫學大學通識教育中心
807 高雄市三民區十全一路 100 號
吳炎崑、盧永毅
萬能科技大學工業管理系

摘要

本文主要探討一種特殊情況下的連續型線性規劃問題，簡稱為(SP)問題。針對此一問題，本文提出一種有效率的方法來找尋(SP)問題的近似最佳解及近似最佳值，此法只需要利用遞迴關係求解一個有限維的線性規劃問題，即可找出原問題的近似最佳解及近似最佳值。本文就(SP)問題的求解程序提出一個演算法，此演算法不僅能容易的求解(SP)問題，並能估算近似值的誤差。最後列舉例子來證實本文所提演算法的可行性。

關鍵詞：連續型線性規劃問題，遞迴方法
(*聯絡人: cfwen@cc.kmu.edu.tw)